MIDTERM: BMATH TOPOLOGY

Date: **27th February 2020**
The Total points is **108** and the maximum you can score is **100** points.

(1) (7+7+7+7=28 points) Answer the following multiple choice questions about each of them. Write all correct options. No justification needed. **No partial credit will be given if a correct option is missing or an incorrect option is written.**

(a) Let A and B be two infinite sets with $|A| \leq |B|$ where $|A|$ denote the cardinality of A. Which of the following statements are true?
  (i) $|A \cup B| = |B|$.
  (ii) $|A \times B| > |B|$.
  (iii) $|A^B| > |B|$.
  (iv) $|A^B| > |A \times B|$
(b) Let $(X, \mathscr{T})$ be a topological space. Which of the following are true?
  (i) If $\mathscr{B}_1$ and $\mathscr{B}_2$ are basis of $\mathscr{T}$ then $\mathscr{B}_1 \cup \mathscr{B}_2$ is a basis of $\mathscr{T}$.
  (ii) If $\mathscr{B}_1$ and $\mathscr{B}_2$ are basis of $\mathscr{T}$ then $\mathscr{B}_1 \cap \mathscr{B}_2$ is a basis of $\mathscr{T}$.
  (iii) Let $\mathscr{B}_1$ be a basis of $\mathscr{T}$ and $\mathscr{B}'$ be a basis of a topology $\mathscr{T}'$ on X. If $\mathscr{T}'$ is finer that $\mathscr{T}$ then $\mathscr{B}'$ contains $\mathscr{B}$.
  (iv) If $\mathscr{B}$ is a basis of X and Y is a subspace of X then the collection $\{B \cap Y : B \in \mathscr{B}\}$ is a basis of Y.
(c) Let $f : X \to Y$ be a function between topological spaces. Which of the following are true?
  (i) If X has discrete topology then f is continuous.
  (ii) If Y has discrete topology then f is continuous.
  (iii) If f is bijective and continuous then X is hausdorff if and only if Y is hausdorff.
  (iv) If Y is discrete and f is bijective and continuous X is discrete.
(d) Let X and Y be topological spaces.
  (i) If X is Hausdorff and Y is a subspace then Y is Hausdorff.
  (ii) If X and Y are Hausdorff then $X \times Y$ is Hausdorff.
  (iii) If Y is a quotient space of X and X is Hausdorff then Y is Hausdorff.
  (iv) If X and Y satify T_1 axiom then $X \times Y$ satisfy T_1 axiom.

(2) (4+8=12 points) Let X be a topological space and A be a subset of X. Define an accumulation point of A. Let A' denote the set of accumulation points of A. Prove or disprove A' is a closed set.

(3) (20 points) Consider $\mathbb{R}^2$ with lexicographic ordering ($(1,2) < (2,1)$). When is a subset U of $\mathbb{R}^2$ open in the order topology? Let π_1 and π_2 be the first and second projection maps to $\mathbb{R}$. Let $i_1(x) = (x, 0)$ and $i_2(x) = (0, x)$ be maps from $\mathbb{R}$ to $\mathbb{R}^2$. When $\mathbb{R}$ is given the standard topology and R^2 the order topology which of these maps are continuous? Which among those are open?

(4) (14 points) Let $\mathbb{R}^\omega$ be the product of countable copies of $\mathbb{R}$ and $\mathbb{R}^\infty$ be the subset consisting of sequences (x_n) where only finitely many x_n's are non zero. Compute the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ when $\mathbb{R}^\omega$ has (a) the product topology and (b) the box topology.

(5) (4+5+5=14 points) When is a topological space called first countable? Show that a metric space is a first countable space. Give an example of a first countable space which satify T_1 axiom but is not a metric space?

(6) (4+8+8=20 points) Let $p : X \to Y$ be map of topological spaces. When is p called a quotient map? Let $p : X \to Y$ be a continuous map and $s : Y \to X$ be continuous fuction such that $p \circ s$ is identity map. Show that p is a quotient map.

On $\mathbb{R}^2$ with standard topology, define an equivalence relation as follows:

$$(x, y) \sim (x', y') \text{ if } x^3 + y = x'^3 + y'$$

Let $X = \mathbb{R}^2 / \sim$ be the quotient space. Show that X is homeomorphic to $\mathbb{R}$ with standard topology.